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ON THE STABILITY ANALYSIS OF BOUNDARY CONDITIONS FOR THE WAVE EQUATION BY ENERGY METHODS. PART I: THE HOMOGENEOUS CASE

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ABSTRACT. We reconsider the stability theory of boundary conditions for the wave equation from the point of view of energy techniques. We study, for the case of the homogeneous half-space, a large class of boundary conditions including the so-called absorbing conditions. We show that the results of strong stability in the sense of Kreiss, studied from the point of view of the modal analysis by Trefethen and Halpern, always correspond to the decay in time of a particular energy. This result leads to the derivation of new estimates for the solution of the associated mixed problem.

1. INTRODUCTION AND SUMMARY

The theory of the stability of initial boundary value problems for hyperbolic systems underwent an important development at the beginning of the 1970s with the major work of Kreiss [10]. A very interesting review paper of this theory has been recently given by Higdon in [8].

It appears that the relative complexity of this theory comes from the difficulty of a good definition for the stability or well-posedness of these problems and from the technical character of the proofs. In fact, one can roughly distinguish two kinds of stability definitions, namely

—weak well-posedness (or weak stability),

—strong well-posedness (or strong stability).

Weak well-posedness corresponds to classical well-posedness in the sense of Hadamard, meaning that there is existence and uniqueness of the solution, and that one can estimate some norm (let us say of Sobolev type) of the solution by some norm of the data. This implies, since the equations are linear, that the map data $\rightarrow$ solution is continuous for appropriate topologies. In the definition of strong stability, Kreiss prescribes a priori the norms for which he wants to obtain some estimates, which is of course a stronger result.

For instance, if one considers the Cauchy problem where initial conditions are the only data, one requires that the L^2-norm (in space and time, including the trace on the boundary) of the solution must be estimated (modulo a constant which may depend on the interval of time $[0, T]$ one considers) by the L^2-norm of the initial data: *there is no loss of derivatives.* On the other hand, the

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problem is weakly well-posed as soon as one can bound the same L^2-norm of the solution by some higher Sobolev norm of the data. We shall give a simple concrete illustration of the distinction between these two notions in this article (see §2.2).

In the case of a half-space in $\mathbb{R}^d$, the two types of stability have been studied for first-order strictly hyperbolic systems with constant coefficients via so-called modal analysis, which means that one uses the Laplace transform in time and a Fourier transform with respect to the variables tangent to the boundary of the half-space. This study leads to the concepts of generalized eigenvalues and characteristic equation, which are extensively discussed in [8] with a geometrical interpretation using characteristic manifolds and group velocities. We shall not enter into the details here. Let us simply mention that the characteristic equation takes the form

$$F(s, \theta) = 0, \tag{1.1}$$

where $s \in \mathbb{C}$ is the unknown and θ belongs to the unit sphere of $\mathbb{R}^{d-1}$. Once the characteristic equation (1.1), which is a polynomial equation with respect to s, has been determined, the difficulty is reduced to the location of the solutions of this equation in the complex plane. The case of weakly well-posed problems corresponds to the result of Hersch [5], while the case of strongly well-posed problems corresponds to the result of Kreiss [10]. These results can be summarized as follows

$$\text{Weak well-posedness} \Leftrightarrow \{\text{solutions of } (1.1)\} \subset \{\operatorname{Re} s \geq 0\},$$
$$\text{Strong well-posedness} \Leftrightarrow \{\text{solutions of } (1.1)\} \subset \{\operatorname{Re} s > 0\}.$$

Otherwise, the corresponding problem is said to be strongly ill-posed. In recent years, the emphasis has been put on strong stability and the works of Kreiss have been generalized to various situations by different authors (see [8, 12] for instance). One reason is probably the fact that this concept has led to an analogous stability theory for finite difference approximations to mixed hyperbolic initial boundary value problems (see [4, 3, 13, 14]). The other major interest of the concept of strong well-posedness is the fact that Kreiss was unable to prove that the stability results could be extended to smooth variable coefficients and lower-order perturbations. His proof is very complicated and makes use of the theory of symmetrizers and pseudodifferential operators ([10]). The precise condition about the variations of the coefficients (which are allowed to vary in space and time) can be stated as follows:

> The coefficients are of class C^∞ with respect to the space and time variables and are asymptotically constant at infinity.

Taking into account the finite velocity of propagation of solutions of hyperbolic systems, the constraint that the coefficients are constant at infinity is not really troublesome. The relative weakness of the result lies more in the fact that one cannot say anything in the case where the coefficients are not smooth.

Of course, this is due to the technique of proof, which uses pseudodifferential operators. It is natural to think of other methods, such as energy methods. In most of the systems derived for physical phenomena, one can associate an energy with the solution that can be shown to be conserved in time in the case of pure initial value problem; these are conservative hyperbolic systems

([2, 16]). The acoustic wave equation, in which we are particularly interested in this paper, belongs to this category of systems. The study of the stability of the wave equation associated with various boundary conditions has a very important practical interest if one thinks, for instance, of absorbing boundary conditions. In their second paper on the subject, Engquist and Majda [1] applied the theory of Kreiss to show the strong stability of their boundary conditions. More recently, Trefethen and Halpern [15] considered a completely general class of boundary conditions for the wave equation and, applying the Kreiss theory, obtained explicit necessary and sufficient criteria (concerning the coefficients of the differential operator appearing in the boundary condition) for the strong well-posedness of the corresponding initial boundary value problem. We shall state their precise results in the next section. In connection with this work, a question naturally arises: is it possible to obtain the same results by energy estimates?

This is the question we intend to address in this paper. In his review [8], Higdon raised briefly this question and concluded that, if energy techniques fail, this does not mean that the problem is not well-posed but simply that the energy method is not well suited for it.

As far as we know, the only case where the energy method has been shown to work is the case of first-order absorbing boundary condition. In the present paper, we show that appropriate energy methods lead to the stability result for a class of boundary conditions which is almost the same as the one considered by Trefethen and Halpern in [15]. More precisely, we show, for the model problem of the wave equation in the homogeneous two-dimensional half-space, that the strong stability result is connected with the decay in time of some energy associated with the solution. This energy is not necessarily the physical one. By energy, we mean a quadratic form with respect to homogeneous linear differential operators of a given order applied to the solution, which is equal to 0 if and only if the solution u is identically equal to 0, provided it vanishes at infinity. More precisely, we show that, if one considers the problem

$$(1.2)\qquad \begin{aligned} &\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0, && x \in \mathbb{R},\ y < 0,\ t > 0, \\ &B_N u = 0, && x = 0,\ t > 0, \\ &u(x, y, 0) = u_0(x, y), && \\ &\frac{\partial u}{\partial t}(x, y, 0) = u_1(x, y), && x = 0,\ y < 0, \end{aligned}$$

where $B_N = B_N(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ is an appropriate homogeneous differential operator of order N, then strong well-posedness of (1.2) will correspond to the decay in time of an energy involving Nth-order derivatives of the solution. For instance, the physical energy

$$(1.3)\qquad E(u; t) = \frac{1}{2} \int_\Omega \left(\left| \frac{\partial u}{\partial t} \right|^2 + |\nabla u|^2 \right) dx\, dy$$

is an energy in our sense for $N = 1$. The class of differential operators for which our analysis is valid is the subclass of the one considered by Trefethen and Halpern, for which the directions x and $-x$ play the same role, in other words, for which $B_N = B_N(\frac{\partial}{\partial t}, \frac{\partial}{\partial y}, \frac{\partial^2}{\partial x^2})$. For larger-dimensional problems, this hypothesis is to be replaced by the invariance of B_N with respect to rotations in the tangential hyperplane; thus $B_N = B_N(\frac{\partial}{\partial t}, \frac{\partial}{\partial y}, \Delta_x)$.

We believe that this particular class is the most interesting one for practical applications. For such particular conditions, Higdon [9] observed that the stability analysis essentially reduces to a 1D analysis. This remark will also be the starting point of our method.

It seems to us that our new approach to the stability theory of boundary conditions associated with the wave equation has the following two advantages:

(1) It gives new insight into the Kreiss theory applied to the wave equation. Moreover, it leads to results which are stronger than those simply deduced from the direct application of Kreiss's theorem in the sense that

- (i) it shows that "all" strongly stable boundary conditions are dissipative for an appropriate norm of the solution;
- (ii) it gives rise to L^∞-estimates with respect to time instead of L^2-estimates;
- (iii) these estimates are uniform in time: the constants involved do not depend on time.

(2) Such a method, applied to variable coefficients, leads to strong stability results even when these coefficients are not smooth.

It is point (1) that we develop in the present paper. We shall consider point (2) in a companion paper (part II).

The outline of this article is as follows. In §2, we present the basic ideas of our method applied to rather simple examples of second-order (§2.2) and third-order (§2.3) boundary conditions. In each case, we first treat as a model problem the absorbing boundary condition of Engquist and Majda, and then extend the result to more general cases. We have chosen to develop this section for pedagogical reasons since the ideas for the proof in the general case are not so obvious. Section 3 is devoted to a generalization of the results of §2 to general odd- (§3.1) and even-order conditions (§3.2).

2. Energy estimates for the wave equation in a half-space: low-order boundary conditions

As stated in §1, we consider the wave equation in a 2D homogeneous medium and assume for simplicity that the propagation velocity is equal to 1,

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0, \qquad \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \tag{2.1}$$

The domain of propagation is the half-space

$$\Omega = \{(x, y) \in \mathbb{R}^2, \ x \in \mathbb{R}, \ y < 0\},$$

and we shall denote its boundary by $\Gamma = \partial\Omega$. The wave equation will be associated with initial conditions

$$\begin{aligned} u(x, y, 0) &= u_0(x, y), \\ \frac{\partial u}{\partial t}(x, y, 0) &= u_1(x, y), \end{aligned} \qquad \text{in } \Omega, \tag{2.2}$$

which will be the only data of the problem, and with the boundary condition

$$Bu = B\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) u = 0 \quad \text{on } \Gamma, \tag{2.3}$$

where B denotes a homogeneous linear differential operator involving *even-order derivatives only* with respect to the *tangential variable* x.

In this section, we consider the cases where B is of order 1, 2, or 3, to describe the basic principle of our method. The case of first-order condition is simple and classical (§2.1). For second-order and third-order boundary conditions (§§2.2 and 2.3) we begin by considering the classical absorbing boundary conditions of Engquist and Majda and show that their stability can be proved by energy methods. Then we consider the cases of more general second-order or third-order differential operators and show that the energy method leads to exactly the same conditions as the ones obtained by directly applying the results of Trefethen and Halpern. For the case of second-order boundary conditions we illustrate in a particular example the difference between strong and weak well-posedness.

The calculations we make in this section are formal. Our goal is to obtain a priori estimates on the solution u, if one assumes that this solution exists, is unique, and is sufficiently regular to justify the technical manipulations we shall be led to do. All this approach can be justified a posteriori by means of techniques of functional analysis. In fact, almost all our computations are based on the following well-known identity:

(2.4) If v denotes a sufficiently smooth solution of the wave equation $\partial^2 v/\partial t^2 - \Delta v = 0$ in the domain $\{x \in \mathbb{R},\ y < 0,\ t > 0\}$, one has

$$\frac{d}{dt}E(v\,;\,t) = \int_\Gamma \frac{\partial v}{\partial y}\frac{\partial v}{\partial t}\,dx,$$

where we have set $\Omega = \{(x, y) \mid x \in \mathbb{R},\ y < 0\}$ and $\Gamma = \partial\Omega$, and where $E(v\,;\,t)$ is defined by (1.3).

2.1. **First-order boundary conditions.** Here we consider the boundary condition

$$Bu = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial y} = 0.$$

In this case, it is well known that applying (2.4) to the function u itself leads to the identity

$$\frac{d}{dt}\left\{\frac{1}{2}\int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 dx\,dy + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx\,dy\right\} + \int_\Gamma \left|\frac{\partial u}{\partial t}\right|^2 dx = 0, \tag{2.5}$$

which shows that the "first-order energy", which coincides with the physical energy defined by (1.3), is a decreasing function of time. The well-posedness of the initial boundary value problem follows immediately. More precisely, one shows that if the energy $E_1(0)$ is finite, which means that $(u_0, u_1) \in H^1(\Omega) \times L^2(\Omega)$, then (1.2) has a unique weak solution (i.e., in the sense of distributions, see [11] for instance) which satisfies

$$u \in W^{1,\infty}(\mathbb{R}^+\,;\,L^2(\Omega)) \cap L^\infty(\mathbb{R}^+\,;\,H^1(\Omega)), \tag{2.6}$$

with the estimates

$$\begin{aligned} \left\|\frac{\partial u}{\partial t}\right\|_{L^\infty(\mathbb{R}\,;\,L^2(\Omega))} &\le C(\|\nabla u_0\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)}), \\ \|\nabla u\|_{L^\infty(\mathbb{R}^+\,;\,L^2(\Omega))} &\le C(\|\nabla u_0\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)}). \end{aligned} \tag{2.7}$$

Note that such estimates correspond to a strong stability result in the sense of Kreiss, since first-order derivatives of the solutions can be estimated at a time t by the same derivatives evaluated at $t = 0$. In fact, the result we obtain here is even stronger than the one one gets by a simple and direct application of the Kreiss theory, since we get here uniform estimates with respect to time (if one uses L^2 norms with respect to the space variables) instead of the $L^2(0, T; L^2(\Omega))$ estimate given by the Kreiss theory. Moreover, as in the Kreiss theory, we obtain an estimate of the trace of the solution on the boundary (Γ) since, thanks to (2.5), we have

$$\int_0^{+\infty} \left|\frac{\partial u}{\partial t}\right|^2_{L^2(\Gamma)} dt \leq \frac{1}{2}(\|u_1\|^2_{L^2(\Omega)} + \|\nabla u_0\|^2_{L^2(\Omega)}), \tag{2.8}$$

and also, using the boundary condition,

$$\int_0^{+\infty} \left|\frac{\partial u}{\partial y}\right|^2_{L^2(\Gamma)} dt \leq \frac{1}{2}(\|u_1\|^2_{L^2(\Omega)} + \|\nabla u_0\|^2_{L^2(\Omega)}). \tag{2.9}$$

2.2. Second-order boundary conditions.

2.2.1. *The classical condition.* We consider now the boundary condition

$$Bu = \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial y \partial t} - \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = 0, \tag{2.10}$$

which is known [1] to lead to a strongly well-posed problem. A natural question is: what form does this well-posedness take if one uses energy techniques instead of normal mode analysis? In fact, it is not clear that with the condition (2.10), the "first-order energy" $E_1(u; t)$ is a decreasing function of time. Nevertheless, we shall see that such a decay occurs for another energy, which will be a "second-order energy" in the sense of a positive quadratic form involving second-order derivatives of the solution.

To obtain this result, we observe that, because of the fact that only the second-order derivative in x occurs in the expression of Bu, the boundary condition (2.10) can be rewritten, as soon as the solution we consider is sufficiently regular, only with derivatives with respect to y and t. This was previously observed by Higdon in [9]. By substituting $\partial^2 u/\partial t^2 - \partial^2 u/\partial y^2$ for $\partial^2 u/\partial x^2$, we obtain

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\right)^2 u = \frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial t \partial y} + \frac{\partial^2 u}{\partial y^2} = 0,$$

which we can also write

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial y^2} = -2\frac{\partial^2 u}{\partial y \partial t}. \tag{2.11}$$

Now let us note that, as we are in the constant-coefficient case, if u is a smooth solution of the wave equation, so are $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial y}$. Therefore, we can apply (2.4) successively to $v = \frac{\partial u}{\partial t}$ and $v = \frac{\partial u}{\partial y}$ to obtain the two following equalities:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega \left(\left|\frac{\partial^2 u}{\partial t^2}\right|^2 + \left|\frac{\partial^2 u}{\partial x \partial t}\right|^2 + \left|\frac{\partial^2 u}{\partial y \partial t}\right|^2\right) dx\, dy\right\} = \int_\Gamma \frac{\partial^2 u}{\partial t^2}\frac{\partial^2 u}{\partial y \partial t}\, dx, \\ &\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega \left(\left|\frac{\partial^2 u}{\partial t \partial y}\right|^2 + \left|\frac{\partial^2 u}{\partial x \partial y}\right|^2 + \left|\frac{\partial^2 u}{\partial y^2}\right|^2\right) dx\, dy\right\} = \int_\Gamma \frac{\partial^2 u}{\partial t \partial y}\frac{\partial^2 u}{\partial y^2}\, dx. \end{aligned} \tag{2.12}$$

If we add these two equations term by term, it is natural to introduce the following "*second-order energy*":

$$E_2(u;t) = \frac{1}{2}\int_\Omega \left(\left|\frac{\partial^2 u}{\partial t^2}\right|^2 + \left|\frac{\partial^2 u}{\partial x \partial t}\right|^2 + 2\left|\frac{\partial^2 u}{\partial t \partial y}\right|^2 + \left|\frac{\partial^2 u}{\partial x \partial y}\right|^2 + \left|\frac{\partial^2 u}{\partial y^2}\right|^2 \right) dx\, dy.$$

So we have

$$\frac{dE_2}{dt} = \int_\Gamma \left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial y^2}\right) \frac{\partial^2 u}{\partial y \partial t}\, dx,$$

that is to say, by (2.11),

$$\frac{dE_2}{dt} = -2\int_\Gamma \left|\frac{\partial^2 u}{\partial y \partial t}\right|^2 dx, \tag{2.13}$$

which means that $E_2(u;t)$ is a decreasing function of time. In particular, it remains finite provided

$$(u_0, u_1) \in H^2(\Omega) \times H^1(\Omega). \tag{2.14}$$

In this case, $E_2(u;t)$ can be uniformly estimated with the help of $\|u_0\|^2_{H^2(\Omega)} + \|u_1\|^2_{H^1(\Omega)}$. This gives a uniform estimate of all second-order derivatives of u except $\partial^2 u/\partial x^2$, and this last estimate follows from the fact that, because of the wave equation, $\partial^2 u/\partial x^2 = \partial^2 u/\partial t^2 - \partial^2 u/\partial y^2$.

These estimates enable us to prove the existence of a weak solution (in a sense that we shall make precise in the next section) of (1.2) satisfying

$$u \in W^{2,\infty}(0, T; L^2(\Omega)) \cap W^{1,\infty}(\mathbb{R}^+; H^1(\Omega)) \cap L^\infty(\mathbb{R}^+; H^2(\Omega)) \tag{2.15}$$

if (2.14) holds. Moreover, the estimate

$$\|D^2 u\|_{L^\infty(\mathbb{R}^+; L^2(\Omega))} \le C(\|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^1(\Omega)}), \tag{2.16}$$

valid for any second-order derivative of u, shows that we have proven a strong stability result (no loss of derivatives). These interior estimates are again completed by L^2-type estimates on the boundary. Indeed, from (2.13) we easily deduce

$$\int_0^{+\infty} \left|\frac{\partial^2 u}{\partial y \partial t}\right|^2_{L^2(\Gamma)} dt \le C(\|u_0\|^2_{H^2(\Omega)} + \|u_1\|^2_{H^1(\Omega)}). \tag{2.17}$$

This is not the only boundary estimate one can obtain, since for instance (2.10) directly provides an estimate of the quantity $\partial^2 u/\partial t^2 - \frac{1}{2}(\partial^2 u/\partial x^2)$. Nevertheless, both a priori estimates (2.16) and (2.17) are sufficient to give a meaning to u as a weak solution of problem (1.2). (See definition (2.25) in the next subsection.)

Note that for second-order boundary conditions, we need more regularity in the initial data than for a first-order condition, but we also get more regularity in the solution. This is not surprising since the operator B involves higher derivatives. One could wonder how to give a sense to a solution of the same problem when we assume only, for instance, that $(u_0, u_1) \in H^1(\Omega) \times L^2(\Omega)$, which means that only the first-order energy $E(u, t)$ is finite at $t = 0$. This should be possible thanks to a duality process, but we shall not examine this point in the present work.

2.2.2. *More general second-order boundary conditions.* In the theory of absorbing boundary conditions for the wave equation ([1, 15, 9]), the condition (2.10) corresponds to a second-order Taylor approximation of the function $\sqrt{1-x^2}$ (which comes from the symbol of the Dirichlet-Neumann operator for the wave equation). More generally, one can consider approximations of the form

$$\sqrt{1-x^2} \simeq \gamma - \beta x^2, \qquad (\beta, \gamma) \in \mathbb{R}^2, \; \beta \neq 0, \tag{2.18}$$

leading to the boundary condition

$$\gamma \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial y \, \partial t} - \beta \frac{\partial^2 u}{\partial x^2} = 0. \tag{2.19}$$

This is the moment for us to recall the results obtained by Trefethen and Halpern in [15], which concern the well-posedness of the initial boundary value problem obtained by coupling the wave equation to the boundary condition $Bu = 0$ derived from the approximation of the function $\sqrt{1-x^2}$ by a general rational function

$$\sqrt{1-x^2} \cong R(x) = \frac{P(x)}{Q(x)}, \tag{2.20}$$

where $P(x)$, $Q(x)$ are polynomials. They recall the following

Theorem A [15]. *The intial boundary value problem* (1.2) *corresponding to the boundary condition* $Bu = 0$ *associated with the approximation* (2.20) *is strongly well posed if and only if the two following conditions are satisfied*:

(i) *All the poles and zeros of the rational function* $\frac{R(x)}{x}$ *are real and interlace along the real axis*;

(ii) $R(x) > 0$ *for* $-1 \leq x \leq 1$.

Remark. Condition (i) implies in particular that

$$0 \leq \deg P - \deg Q \leq 2.$$

Let us apply this criterion to the condition (2.19) where $R(x) = \gamma - \beta x^2$. It is immediate that (i) holds if and only if $\gamma\beta > 0$, while criterion (ii) and (2.20) imply $\gamma > 0$ and $\gamma - \beta > 0$. Therefore, problem (1.2) with boundary condition (2.19) is strongly well-posed if and only if

$$0 < \beta < \gamma. \tag{2.21}$$

Let us see now how this condition appears when one uses energy techniques. Of course, we still can use the two equalities (2.12), which are independent of the boundary condition. Now, instead of summing these two equalities term by term, we multiply the first of them by $\gamma - \beta$ and the second by β and add the two results. We obtain

$$\frac{dE_2}{dt} = \int_\Gamma \left((\gamma - \beta) \frac{\partial^2 u}{\partial t^2} + \beta \frac{\partial^2 u}{\partial y^2} \right) \frac{\partial^2 u}{\partial y \, \partial t} \, dx, \tag{2.22}$$

where the function $E_2(u; t)$ is defined by

$$\begin{aligned} E_2(u; t) = {} & \frac{\gamma - \beta}{2} \int_\Omega \left(\left| \frac{\partial^2 u}{\partial t^2} \right|^2 + \left| \frac{\partial^2 u}{\partial x \, \partial t} \right|^2 + \left| \frac{\partial^2 u}{\partial y \, \partial t} \right|^2 \right) dx \, dy \\ & + \frac{\beta}{2} \int_\Omega \left(\left| \frac{\partial^2 u}{\partial y \, \partial t} \right|^2 + \left| \frac{\partial^2 u}{\partial x \, \partial y} \right|^2 + \left| \frac{\partial^2 u}{\partial y^2} \right|^2 \right) dx \, dy. \end{aligned} \tag{2.23}$$

Now replacing $\partial^2u/\partial x^2$ by $(\partial^2u/\partial t^2 - \partial^2u/\partial y^2)$, we observe that (2.19) is equivalent to

$$(\gamma-\beta)\frac{\partial^2u}{\partial t^2}+\beta\frac{\partial^2u}{\partial y^2}=-\frac{\partial^2u}{\partial y\,\partial t},$$

so that (2.22) becomes

$$\frac{dE_2}{dt}=-\int_\Gamma\left|\frac{\partial^2u}{\partial y\,\partial t}\right|^2 dx. \tag{2.24}$$

This proves that the function $E_2(u;t)$ is decaying in time. To be able to deduce a priori estimates on the solution, we need that $E_2(u;t)$ be a "second-order energy", i.e., a positive quadratic form. This implies $0\le\beta\le\gamma$. Now if we want to estimate all the second-order derivatives of u, which is necessary to get the strong stability property, we see that condition (2.21) must be satisfied. Conversely, if (2.21) holds, it is then easy, mimicking what we did in §2.2.1, to obtain the estimates (2.16) and (2.17) and then to prove the existence of a unique weak solution of u, to be understood in the following sense:

- $u(x,y,t)$ satisfies (2.15),
- $\forall v(x,y)\in V=\{v\in H^1(\Omega)/v|_\Gamma\in H^1(\Gamma)\}$

$$\begin{aligned}&\frac{d^3}{dt^3}\left(\int_\Omega uv\,dx\,dy\right)+\gamma\frac{d^2}{dt^2}\left(\int_\Gamma uv\,dx\right)\\&\quad+\frac{d}{dt}\left(\int_\Omega\nabla u\cdot\nabla v\,dx\,dy\right)+\beta\int_\Gamma\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,dx=0\quad\text{in }\mathscr{D}'(0,T),\end{aligned}\tag{2.25}$$

- $u(x,y,0)=u_0(x,y)$; $\frac{\partial u}{\partial t}(x,y,0)=u_1(x,y)$.

We now state our result in the following theorem:

Theorem 2.1. *Under the condition* (2.21), *the initial boundary value problem* (2.1), (2.19) *is strongly well posed in the sense of Kreiss, and the unique solution satisfies the identity* (2.24), *which yields the decay in time of the second-order energy* E_2 *defined by* (2.23). *Moreover, this solution satisfies the a priori estimate*

$$\sup_{|\alpha|=2}\|D^\alpha u\|_{L^\infty(\mathbb{R}^+;L^2(\Omega))}\le C(\|u_0\|_{H^2(\Omega)}+\|u_1\|_{H^1(\Omega)}),$$

where the positive constant C *depends only on* γ *and* β.

Remark. If one does not look for a strong stability result in the sense of Kreiss but only for a well-posedness result in the sense of Hadamard, it is sufficient to apply the condition of Hersch [5]. In our particular case, this condition implies simply that

$$\gamma>0,\qquad\beta>0.$$

This means that for $0<\gamma\le\beta<+\infty$, problem (1.2) is only weakly well-posed. This fact can also be obtained by using energy techniques to get a priori estimates. Indeed, let u be a regular solution of (1.2), and introduce

$$v=\gamma\frac{\partial^2u}{\partial t^2}+\frac{\partial^2u}{\partial y\,\partial t}-\beta\frac{\partial^2u}{\partial x^2}. \tag{2.26}$$

It is clear that v is solution of

$$\frac{\partial^2 v}{\partial t^2} - \Delta v = 0 \quad \text{in } \Omega, \qquad v|_\Gamma = 0 \quad \text{on } \Gamma, \tag{2.27}$$

which is a well-posed problem. Now if we want to get estimates on u, we have only to check that the initial value problem

$$\begin{aligned} &\gamma\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial y \partial t} - \beta\frac{\partial^2 u}{\partial x^2} = v, \qquad (x, y) \in \Omega, \ t > 0, \\ &u(x, 0) = u_0(x, y), \qquad \frac{\partial u}{\partial t}(x, 0) = u_1(x, y) \end{aligned} \tag{2.28}$$

is well posed. But multiplying equation (2.27) by $\frac{\partial u}{\partial t}$ and integrating over Ω gives

$$\frac{d}{dt}\left\{\frac{\gamma}{2}\int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 dx\,dy + \frac{\beta}{2}\int_\Omega \left|\frac{\partial u}{\partial x}\right|^2 dx\,dy\right\} + \int_\Gamma \left|\frac{\partial u}{\partial t}\right|^2 dx = \int_\Omega v\frac{\partial u}{\partial t}\,dx\,dy,$$

which easily leads to the following estimates in the interval $[0, T]$ (use Gronwall's Lemma):

$$\begin{aligned} \left\|\frac{\partial u}{\partial t}\right\|_{L^\infty(0,T;L^2)} &\le C(T)\|v\|_{L^2(0,T;L^2)}, \\ \left\|\frac{\partial u}{\partial x}\right\|_{L^\infty(0,T;L^2)} &\le C(T)\|v\|_{L^2(0,T;L^2)}. \end{aligned} \tag{2.29}$$

Moreover, multiplying equation (2.27) by $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial y}$ and integrating over Ω, we get

$$\begin{aligned} \frac{d}{dt}&\left\{\frac{\gamma}{2}\int_\Omega \left|\frac{\partial u}{\partial t} + \frac{\partial u}{\partial y}\right|^2 dx\,dy + \frac{\beta}{2}\int_\Omega \left|\frac{\partial u}{\partial x}\right|^2 dx\,dy\right\} + \int_\Gamma \left|\frac{\partial u}{\partial x}\right|^2 dx \\ &= \int_\Omega v\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial y}\right) dx\,dy, \end{aligned}$$

which leads to the estimate

$$\left\|\frac{\partial u}{\partial y}\right\|_{L^\infty(0,T;L^2)} \le C(T)\|v\|_{L^2(0,T;L^2)}. \tag{2.30}$$

We need now an estimate of the function v of (2.27). This problem can be solved explicitly, using the theory of images for the Dirichlet condition. Then, by a Fourier transform in space, one easily checks that

$$\|v(t)\|^2_{L^2(\Omega)} \le 2\{\|v_0\|^2_{L^2(\Omega)} + \|v_1\|^2_{H^{-1}(\Omega)}\}, \tag{2.31}$$

where $v(x, y, 0) = v_0(x, y)$ and $\frac{\partial v}{\partial t}(x, y, 0) = v_1(x, y)$. But, by the definition of v, we have

$$\begin{aligned} v_0 &= \gamma\frac{\partial^2 u_0}{\partial x^2} + (\gamma - \beta)\frac{\partial^2 u_0}{\partial y^2} + \frac{\partial u_1}{\partial y}, \\ v_1 &= \frac{\partial^2 u_1}{\partial x^2} + (\gamma - \beta)\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial}{\partial y}\Delta u_0, \end{aligned}$$

so that

$$\|v_0\|^2_{L^2(\Omega)} + \|v_1\|^2_{H^{-1}(\Omega)} \le C\{\|u_0\|^2_{H^2(\Omega)} + \|u_1\|^2_{H^1(\Omega)}\}. \tag{2.32}$$

Regrouping (2.29), (2.30), (2.31), and (2.32), we obtain

$$\|Du\|_{L^\infty(0,T;L^2)} \le C(T)\{\|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^1(\Omega)}\}. \tag{2.33}$$

This proves the well-posedness of the initial boundary problem as soon as $\beta, \gamma > 0$. Nevertheless, this result corresponds only to a weak stability result since we lose one order of derivative in our estimates.

2.3. **Third-order boundary conditions.**

2.3.1. *The classical condition.* We consider now the classical third-order condition

$$\frac{\partial^3 u}{\partial t^3} + \frac{\partial^3 u}{\partial t^2 \partial y} - \frac{3}{4}\frac{\partial^3 u}{\partial t \partial x^2} - \frac{1}{4}\frac{\partial^3 u}{\partial x^2 \partial y} = 0, \tag{2.34}$$

which is known [9], at least for smooth solutions, to be equivalent to $(\frac{\partial}{\partial t} + \frac{\partial}{\partial y})^3 u = 0$, which we can also write

$$\frac{\partial^3 u}{\partial t^3} + \frac{\partial^3 u}{\partial t^2 \partial y} + \frac{\partial^3 u}{\partial t \partial y^2} + \frac{\partial^3 u}{\partial y^3} = -2\left(\frac{\partial^3 u}{\partial t^2 \partial y} + \frac{\partial^3 u}{\partial t \partial y^2}\right). \tag{2.35}$$

The idea is now to apply the identity (2.4) to the two functions

$$v_1 = \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial t \partial y}, \qquad v_2 = \frac{\partial^2 u}{\partial t \partial y} + \frac{\partial^2 u}{\partial y^2}$$

which are particular solutions of the wave equation if u is also a solution. Moreover, they have the property that $\partial v_1/\partial y = \partial v_2/\partial t$. We thus have

$$\begin{aligned} \frac{d}{dt}E(v_1) &= \int_\Gamma \left(\frac{\partial^3 u}{\partial t^3} + \frac{\partial^3 u}{\partial t^2 \partial y}\right)\left(\frac{\partial^3 u}{\partial t^2 \partial y} + \frac{\partial^3 u}{\partial t \partial y^2}\right) dx, \\ \frac{d}{dt}E(v_2) &= \int_\Gamma \left(\frac{\partial^3 u}{\partial t^2 \partial y} + \frac{\partial^3 u}{\partial t \partial y^2}\right)\left(\frac{\partial^3 u}{\partial t \partial y^2} + \frac{\partial^3 u}{\partial y^3}\right) dx. \end{aligned} \tag{2.36}$$

Summing these two equalities leads to

$$\begin{aligned} &\frac{d}{dt}\{E(v_1) + E(v_2)\} \\ &\quad = \int_\Gamma \left(\frac{\partial^3 u}{\partial t^2 \partial y} + \frac{\partial^3 u}{\partial t \partial y^2}\right)\left(\frac{\partial^3 u}{\partial t^3} + \frac{\partial^3 u}{\partial t^2 \partial y} + \frac{\partial^3 u}{\partial t \partial y^2} + \frac{\partial^3 u}{\partial y^3}\right) dx, \end{aligned}$$

that is, by (2.35),

$$\frac{d}{dt}\{E(v_1) + E(v_2)\} = -2\int_\Gamma \left|\frac{\partial^3 u}{\partial t^2 \partial y} + \frac{\partial^3 u}{\partial t \partial y^2}\right|^2 dx, \tag{2.37}$$

which means that the quadratic form $E(v_1) + E(v_2)$ is a decreasing function of time. Nevertheless, the estimates we can deduce from this fact do not permit us to obtain a strong stability result, since we do not bound all the third-order derivatives of u, but only some linear combinations of them. Therefore, we have to go into the second step of our estimates, which consists in applying the identity (2.4) to the function $v = \partial^2 u/\partial t\,\partial y$. By this process, we obtain

$$\frac{d}{dt}\left\{\frac{1}{2}\int_\Omega\left(\left|\frac{\partial^3 u}{\partial t^2\,\partial y}\right|^2+\left|\frac{\partial^3 u}{\partial t\,\partial x\,\partial y}\right|^2+\left|\frac{\partial^3 u}{\partial t\,\partial y}\right|^2\right)dx\,dy\right\} = \int_\Gamma \frac{\partial^3 u}{\partial t\,\partial y^2}\frac{\partial^3 u}{\partial y\,\partial t^2}\,dx. \tag{2.38}$$

Noticing that $2(a+b)^2-ab=\frac{3}{2}(a+b)^2+\frac{1}{2}(a^2+b^2)$, we see that, if we introduce the third-order energy

$$E_3(u)=E\left(\frac{\partial^2 u}{\partial t\,\partial y}\right)+E\left(\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial t\,\partial y}\right)+E\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial t\,\partial y}\right), \tag{2.39}$$

the addition of (2.37) and (2.38) leads to

$$\frac{dE_3}{dt}(u;t)+\frac{3}{2}\int_\Gamma\left|\frac{\partial^3 u}{\partial t\,\partial y^2}+\frac{\partial^3 u}{\partial t^2\,\partial y}\right|^2 dx + \frac{1}{2}\int_\Gamma\left(\left|\frac{\partial^3 u}{\partial t\,\partial y^2}\right|^2+\left|\frac{\partial^3 u}{\partial t^2\,\partial y}\right|^2\right)dx=0, \tag{2.40}$$

which proves that the energy $E_3(u;t)$ decays in time. In particular, if it is finite at $t=0$, i.e., if one has

$$(u_0, u_1)\in H^3(\Omega)\times H^2(\Omega), \tag{2.41}$$

it remains uniformly bounded in time. From (2.40), we easily see that we get a priori estimates of the L^2-norms of all the third-order derivatives of u except the ones containing second- or third-order derivatives with respect to x. Analogous estimates for these quantities can be derived from the wave equation, which we differentiate in t, x, and y to obtain

$$\begin{aligned}\frac{\partial^3 u}{\partial t\,\partial x^2}&=\frac{\partial^3 u}{\partial t^3}-\frac{\partial^3 u}{\partial t\,\partial y^2},\\ \frac{\partial^3 u}{\partial y\,\partial x^2}&=\frac{\partial^3 u}{\partial t^2\,\partial y}-\frac{\partial^3 u}{\partial y^3},\\ \frac{\partial^3 u}{\partial x^3}&=\frac{\partial^3 u}{\partial t^2\,\partial x}-\frac{\partial^3 u}{\partial x\,\partial y^2}.\end{aligned} \tag{2.42}$$

As these estimates are uniform in time, we can write for any linear third-order differential operator

$$\|D^3 u\|_{L^\infty(\mathbb{R}^+;L^2(\Omega))}\le C(\|u_0\|_{H^3}+\|u_1\|_{H^2}), \tag{2.43}$$

which establishes the strong stability of the initial boundary value problem (1.2) with the boundary condition (2.34).

As in the second-order case, we get L^2-estimates on the boundary Γ, since from (2.39) we directly obtain that

$$\int_0^{+\infty}\int_\Gamma\left(\left|\frac{\partial^3 u}{\partial t\,\partial y^2}\right|^2+\left|\frac{\partial^3 u}{\partial t^2\,\partial y}\right|^2\right)dx\,dt\le C\{\|u_0\|_{H^3}^2+\|u_1\|_{H^2}^2\}, \tag{2.44}$$

which is the equivalent of the estimate (2.17) in the second-order case.

2.3.2. *General third-order boundary conditions.* Condition (2.34) follows from the Padé approximation of the function $\sqrt{1-x^2}$,

$$\sqrt{1-x^2} \approx 1 - \frac{\frac{1}{2}x^2}{1-\frac{1}{4}x^2}. \tag{2.45}$$

Let us generalize this condition by considering the following class of rational approximations of $\sqrt{1-x^2}$:

$$\sqrt{1-x^2} \simeq \gamma - \frac{\beta x^2}{1-\alpha x^2}, \qquad (\alpha, \beta, \gamma) \in \mathbb{R}^3, \tag{2.46}$$

leading to the general third-order condition

$$\gamma \frac{\partial^3 u}{\partial t^3} + \frac{\partial^3 u}{\partial t^2 \partial y} - (\alpha+\beta)\frac{\partial^3 u}{\partial x^2 \partial t} - \alpha \frac{\partial^3 u}{\partial x^2 \partial y} = 0. \tag{2.47}$$

We wish to analyze the stability of this condition using energy techniques. For the sake of simplicity, we shall restrict ourselves to the case $\gamma = 1$. The extension to the general case can be done without any difficulty. We thus consider the boundary condition

$$\frac{\partial^3 u}{\partial t^3} + \frac{\partial^3 u}{\partial t^2 \partial y} - (\alpha+\beta)\frac{\partial^3 u}{\partial x^2 \partial t} - \alpha \frac{\partial^3 u}{\partial x^2 \partial y} = 0, \qquad (\alpha, \beta) \in \mathbb{R}_{\neq 0} \times \mathbb{R}_{\neq 0}. \tag{2.48}$$

If we refer to Theorem A to analyze the strong stability of (2.48), it is easy to see that, for $R(x) = 1 - \beta x^2/(1-\alpha x^2)$, we have
(i) $\Leftrightarrow \alpha > 0$, $\beta + \alpha > 0$,
(ii) $\Leftrightarrow 0 < \beta/(1-\alpha) < 1$.
Therefore, a necessary and sufficient condition for problem (1.2) to be strongly well-posed is

$$\alpha > 0, \quad \beta > 0, \quad 0 < \alpha + \beta < 1. \tag{2.49}$$

Our purpose in this section is to show how these conditions naturally arise when one tries to obtain energy estimates, using a method analogous to the one we have used in §2.2.1.

Since the computations are rather lengthy, we present this approach in several steps.

First step. Consider a real number a and the two functions

$$v_1 = \frac{\partial^2 u}{\partial t^2} + a\frac{\partial^2 u}{\partial t \partial y}, \qquad v_2 = \frac{\partial^2 u}{\partial t \partial y} + a\frac{\partial^2 u}{\partial y^2},$$

where u denotes a (sufficiently smooth) function satisfying the wave equation in Ω and the boundary condition (2.47) on (Γ). Note that v_1 and v_2 are derivatives with respect to t and y of the same function $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial y}$ and are thus related by $\frac{\partial v_1}{\partial y} = \frac{\partial v_2}{\partial t}$. Applying identity (2.4), we obtain

$$\begin{aligned} \frac{d}{dt}\{E(v_1)\} &= \int_\Gamma \left(\frac{\partial^3 u}{\partial t^3} + a\frac{\partial^3 u}{\partial t^2 \partial y}\right)\left(\frac{\partial^3 u}{\partial t^2 \partial y} + a\frac{\partial^3 u}{\partial t \partial y^2}\right) dx, \\ \frac{d}{dt}\{E(v_2)\} &= \int_\Gamma \left(\frac{\partial^3 u}{\partial t^2 \partial y} + a\frac{\partial^3 u}{\partial t \partial y^2}\right)\left(\frac{\partial^3 u}{\partial t \partial y^2} + a\frac{\partial^3 u}{\partial y^3}\right) dx. \end{aligned} \tag{2.50}$$

Therefore, if b denotes a real number, we have

$$\begin{aligned}\frac{d}{dt}\{bE(v_1)+E(v_2)\} &= \int_\Gamma \left(\frac{\partial^3 u}{\partial t^2\partial y}+a\frac{\partial^3 u}{\partial t\,\partial y^2}\right)\\ &\quad\times\left(b\frac{\partial^3 u}{\partial t^3}+ab\frac{\partial^3 u}{\partial t^2\partial y}+\frac{\partial^3 u}{\partial t\,\partial y^2}+a\frac{\partial^3 u}{\partial y^3}\right)dx.\end{aligned}\tag{2.51}$$

Second step. We eliminate the x-derivatives in the boundary condition (2.47), which we rewrite as

$$[1-(\alpha+\beta)]\frac{\partial^3 u}{\partial t^3}+(1-\alpha)\frac{\partial^3 u}{\partial t^2\partial y}+(\alpha+\beta)\frac{\partial^3 u}{\partial t\,\partial y^2}+\alpha\frac{\partial^3 u}{\partial y^3}=0.\tag{2.52}$$

We now choose $a=\lambda\alpha$ and $b=\lambda(1-\alpha-\beta)$, so that, thanks to (2.52),

$$a\frac{\partial^3 u}{\partial y^3}+b\frac{\partial^3 u}{\partial t^3}=-\lambda\left\{(1-\alpha)\frac{\partial^3 u}{\partial t^2\partial y}+(\alpha+\beta)\frac{\partial^3 u}{\partial y^2\partial y}\right\}.$$

Thus, identity (2.51) leads to

$$\begin{aligned}&\frac{d}{dt}\{bE(v_1)+E(v_2)\}\\ &\quad=\int_\Gamma\left(\frac{\partial^3 u}{\partial y\,\partial t^2}+\lambda\alpha\frac{\partial^3 u}{\partial y^2\partial t}\right)\\ &\quad\times\left([\lambda^2\alpha(1-\alpha-\beta)-\lambda(1-\alpha)]\frac{\partial^3 u}{\partial y\,\partial t^2}+[1-\lambda(\alpha+\beta)]\frac{\partial^3 u}{\partial y^2\partial t}\right)dx.\end{aligned}\tag{2.53}$$

Now choose λ so that $\lambda\alpha[\lambda^2\alpha(1-\alpha-\beta)-\lambda(1-\alpha)]=1-\lambda(\alpha+\beta)$, which means that λ is one of the roots of the algebraic equation

$$P(\lambda)=\lambda^3\alpha^2(1-\alpha-\beta)-\lambda^2\alpha(1-\alpha)+\lambda(\alpha+\beta)-1=0.\tag{2.54}$$

In that case, we have

$$\begin{aligned}\frac{d}{dt}\{E(v_2)+bE(v_1)\}&=[\lambda^2\alpha(1-\alpha-\beta)-\lambda(1-\alpha)]\\ &\quad\cdot\int_\Gamma\left|\frac{\partial^3 u}{\partial y\,\partial t^2}+\lambda\alpha\frac{\partial^3 u}{\partial y^2\partial t}\right|^2dx.\end{aligned}\tag{2.55}$$

To deduce appropriate a priori estimates on v_1 and v_2 from (2.55), we have to satisfy

$$b=\lambda(1-\alpha-\beta)>0,\qquad \lambda^2\alpha(1-\alpha-\beta)-\lambda(1-\alpha)\le 0.\tag{2.56}$$

However, this is not sufficient to imply a strong stability result, since the fact that $E(v_1)$ and $E(v_2)$ are bounded does not lead to uniform estimates of all third-order derivatives of u in the domain Ω. To overcome this difficulty, we also apply our basic identity (2.4) to the function $v_3=\partial^2u/\partial y\,\partial t$, exactly as in §2.3.1:

$$\frac{d}{dt}\left\{E\left(\frac{\partial^2 u}{\partial y\,\partial t}\right)\right\}=\int_\Gamma\frac{\partial^2 u}{\partial y\,\partial t^2}\cdot\frac{\partial^2 u}{\partial y^2\partial t}\,dx.\tag{2.57}$$

We multiply (2.57) by some $\delta > 0$ and add the result to (2.55), which gives

$$\frac{d}{dt}\left\{bE(v_1) + E(v_2) + \delta E\left(\frac{\partial^2 u}{\partial y\,\partial t}\right)\right\}$$

$$(2.58)\qquad = [\lambda^2\alpha(1-\alpha-\beta) - \lambda(1-\alpha)]\int_\Gamma \left|\frac{\partial^3 u}{\partial y\,\partial t^2} + \lambda\alpha\frac{\partial^3 u}{\partial y^2\,\partial t}\right|^2 dx$$

$$+\delta\int_\Gamma \frac{\partial^3 u}{\partial y^2\,\partial t}\cdot\frac{\partial^3 u}{\partial y\,\partial t^2}\,dx.$$

In order that the right-hand side of (2.58) be a negative quadratic form, we need that $\lambda^2\alpha(1-\alpha-\beta) - \lambda(1-\alpha) \le 0$, but this is not sufficient. Indeed, let us set

$$(2.59)\qquad \begin{aligned} \delta &= \{\lambda^2\alpha(1-\alpha-\beta) - \lambda(1-\alpha)\}\delta', \\ E_3(u) &= \delta E\left(\frac{\partial^2 u}{\partial y \partial t}\right) + E(v_2) + bE(v_1). \end{aligned}$$

We then have

$$(2.60)\qquad \begin{aligned} \frac{d}{dt}E_3(u) &= \{\lambda^2\alpha(1-\alpha-\beta) - \lambda(1-\alpha)\} \\ &\cdot \int_\Gamma \left\{\left|\frac{\partial^3 u}{\partial y\,\partial t^2} + \lambda\alpha\frac{\partial^3 u}{\partial y^2\,\partial t}\right|^2 + \delta'\frac{\partial^3 u}{\partial y^2\,\partial t}\frac{\partial^3 u}{\partial y\,\partial t^2}\right\} dx. \end{aligned}$$

To be able to complete our task, we must find $\delta' < 0$ such that the quadratic form $(x+\lambda\alpha y)^2 + \delta' xy$ is positive. As the discriminant of this quadratic form is $\Delta = \delta'^2 - \lambda\alpha$, we simply have to ensure that $\Delta < 0$. This implies

$$(2.61)\qquad \lambda\alpha > 0,$$

in which case it suffices to take $\delta' = -\frac{1}{2}\sqrt{\lambda\alpha}$, and the coefficient δ will be strictly positive as soon as

$$(2.62)\qquad \lambda^2\alpha(1-\alpha-\beta) - \lambda(1-\alpha) < 0.$$

Regrouping all our conditions, we see that if one can find a real number λ such that

$$(2.63)\qquad \begin{aligned} &P(\lambda) = \lambda^3\alpha^2(1-\alpha-\beta) - \lambda^2\alpha(1-\alpha) + \lambda(\alpha+\beta) - 1 = 0, \\ &\lambda^2\alpha(1-\alpha-\beta) - \lambda(1-\alpha) < 0, \quad \lambda(1-\alpha-\beta) > 0, \quad \lambda\alpha > 0, \end{aligned}$$

then the third-order energy

$$(2.64)\qquad E_3(u) = bE(v_1) + E(v_2) + \delta E\left(\frac{\partial^2 u}{\partial y \partial t}\right),$$

where (b, δ) are strictly positive numbers, is a decreasing function of time and is thus uniformly bounded in time provided

$$(2.65)\qquad (u_0, u_1) \in H^3(\Omega) \times H^2(\Omega).$$

This leads to uniform estimates in time of the L^2-norms in space of all the third-order derivatives of u except $\partial^3 u/\partial t\,\partial x^2$, $\partial^3 u/\partial x^2\,\partial y$, $\partial^3 u/\partial x^3$. These quantities are easily estimated using the wave equation as in §2.3.1. Finally, if we can find a solution λ of (2.63), we get the following interior estimate:

$$(2.66)\qquad \|D^3 u\|_{L^\infty(\mathbb{R}^+;\,L^2(\Omega))} \le C\{\|u_0\|_{H^3(\Omega)} + \|u_1\|_{H^2(\Omega)}\},$$

which we can complete with boundary estimates, as we have seen in §2.3.1:

$$(2.67)\qquad \begin{aligned} \left\| \frac{\partial^3 u}{\partial y\, \partial t^2} \right\|_{L^2(\mathbb{R}^+\,;\,L^2(\Gamma))} &\le C\{\|u_0\|_{H^3(\Omega)} + \|u_1\|_{H^2(\Omega)}\}\,, \\ \left\| \frac{\partial^3 u}{\partial y^2\, \partial t} \right\|_{L^2(\mathbb{R}^+\,;\,L^2(\Gamma))} &\le C\{\|u_0\|_{H^3(\Omega)} + \|u_1\|_{H^2(\Omega)}\}. \end{aligned}$$

It remains to see what conditions on a and b are sufficient in order that one can find some λ satisfying (2.63). For this, introduce

$$(2.68)\qquad x = \lambda\alpha\,,$$

so that the system (2.63) becomes

$$(2.69)\qquad \begin{gathered} F(x) = [1-(\alpha+\beta)]x^3 - (1-\alpha)x^2 + (\alpha+\beta)x - \alpha = 0\,, \\ x > 0\,, \quad \alpha x(1-(\alpha+\beta)) > 0\,, \quad \alpha x^2(1-(\alpha+\beta)) - \alpha x(1-\alpha) < 0. \end{gathered}$$

The first two inequalities imply

$$(2.70)\qquad \alpha x^2(1-(\alpha+\beta)) > 0.$$

If $\alpha < 0$, then $1-\alpha > 1$ and $\alpha x^2(1-(\alpha+\beta)) - \alpha x(1-\alpha) > 0$, which is impossible because of the third inequality of (2.69). Thus, α is strictly positive, and the first two inequalities of (2.69) are equivalent to

$$(2.71)\qquad x > 0\,, \qquad \alpha+\beta < 1.$$

Moreover, the equation $F(x) = 0$ can be rewritten as

$$(2.72)\qquad [1-(\alpha+\beta)]x^2 - [1-\alpha]x = \frac{\alpha}{x} - (\alpha+\beta)\,,$$

which shows that the last inequality of (2.69) is equivalent to $\frac{\alpha}{x} < \alpha+\beta$. Since $x > 0$, necessarily $\alpha+\beta > 0$, and therefore, we have to find x such that

$$(2.73)\qquad x > \frac{\alpha}{\alpha+\beta}\,, \quad 0 < \alpha+\beta < 1\,, \quad F(x) = 0.$$

It remains to find under what condition $F(x)$ admits a solution in the interval $(\frac{\alpha}{\alpha+\beta}\,, +\infty)$.

From $\alpha+\beta < 1$, we deduce that $\lim_{x\to+\infty} F(x) = +\infty$. On the other hand, since $F(\frac{\alpha}{\alpha+\beta}) = -\beta\alpha^2/(\alpha+\beta)^3$, we can conclude:

(i) If $\beta > 0$, then $F(x)$ admits at least one root in the interval $(\frac{\alpha}{\alpha+\beta}\,, +\infty)$ ($x = 1$ is acceptable since $\frac{\alpha}{\alpha+\beta} < 1$ if $\beta > 0$).

(ii) If $\beta < 0$, then $\frac{\alpha}{\alpha+\beta} > 1$. We remark that

$$F(x) = (x-1)\{[1-(\alpha+\beta)]x^2 - \beta x + \alpha\}\,,$$

which clearly shows that $F(x) > 0$ in the interval $(\frac{\alpha}{\alpha+\beta}\,, +\infty)$.

Finally, our method permits us to obtain a strong stability result if and only if

$$(2.74)\qquad \alpha > 0\,, \quad \beta > 0\,, \quad 0 < \alpha+\beta < 1\,,$$

which are exactly the conditions we obtain by applying the Kreiss criterion. We have proved the following result:

Theorem 2.2. *Under the condition* (2.49), *the initial boundary value problem* (2.1), (2.48) *is strongly well posed in the sense of Kreiss, and the unique solution satisfies the identity* (2.60), *which yields the decay in time of the third-order energy* E_3 *defined by* (2.64). *Moreover, this solution satisfies the a priori estimate*

$$\sup_{|\alpha|=3} \|D^\alpha u\|_{L^\infty(\mathbb{R}^+;L^2(\Omega))} \le C(\|u_0\|_{H^3(\Omega)} + \|u_1\|_{H^2(\Omega)}),$$

where the positive constant C *depends only on* γ *and* β.

It is interesting to note that the equation $F(x) = 0$, which appears naturally in this approach, is nothing but the "*characteristic equation*" that one obtains when applying the normal mode analysis. Indeed, looking for generalized eigenvalues in the form

$$u(x, y, t) = \exp(-st)\exp\xi y \exp ikx, \qquad k \in \mathbb{R}, \ \operatorname{Re}\xi \ge 0 \tag{2.75}$$

leads to the two equations

$$\begin{aligned} &k^2 = \xi^2 - s^2 \quad \text{(interior wave equation)}, \\ &-s^3 + s^2\xi - (\alpha+\beta)k^2 s + \alpha k^2 \xi = 0 \quad \text{(boundary condition)}. \end{aligned} \tag{2.76}$$

Eliminating k^2 between these two equations is equivalent to eliminating the x-derivatives in the boundary condition (2.47). We obtain the *characteristic equation*

$$[1-(\alpha+\beta)]s^3 - (1-\alpha)s^2\xi + (\alpha+\beta)s\xi^2 - \alpha\xi^3 = 0, \tag{2.77}$$

which is exactly $F(x) = 0$ if we set $s = x\xi$.

3. Energy estimates for the wave equation in a half-space: the general case

We now consider general absorbing boundary conditions for (2.1) obtained from the approximation of $\sqrt{1-s^2}$ by the rational function

$$r(s) = \gamma - \sum_{k=1}^{N} \frac{\beta_k s^2}{1-\alpha_k s^2}, \tag{3.1}$$

where α_k, β_k, and γ are real numbers. The resulting boundary condition can be written in the Fourier domain $((x, t) \to (k, \omega))$ in the form

$$\frac{\partial \hat{u}}{\partial y} + i\omega\gamma\hat{u} + i\omega\sum_{k=1}^{N}\frac{\beta_k k^2}{\omega^2 - \alpha_k k^2}\hat{u} = 0. \tag{3.2}$$

Returning to (x, t)-variables, we can reinterpret condition (3.2) as a system

$$\frac{\partial u}{\partial y} + \gamma\frac{\partial u}{\partial t} - \sum \beta_k \frac{\partial \varphi_k}{\partial t} = 0, \quad \frac{\partial^2 \varphi_k}{\partial t^2} - \alpha_k\frac{\partial^2\varphi_k}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} \tag{3.3}$$

or, equivalently, as

$$B_{2N+1}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)u = 0, \tag{3.4}$$

where the linear operator B_{2N+1}, a differential operator of order $2N+1$, is given by

$$B_{2N+1}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \left(\frac{\partial}{\partial y} + \gamma\frac{\partial}{\partial t}\right)\prod_{k=1}^{N}\left(\frac{\partial^2}{\partial t^2} - \alpha_k\frac{\partial^2}{\partial x^2}\right) - \frac{\partial}{\partial t}\sum_{k=1}^{N}\beta_k\frac{\partial^2}{\partial x^2}\left[\prod_{j\neq k}\left(\frac{\partial^2}{\partial t^2} - \alpha_j\frac{\partial^2}{\partial x^2}\right)\right]. \tag{3.5}$$

We can of course apply Theorem A to obtain necessary and sufficient conditions on the coefficients $(\gamma, \alpha_j, \beta_k)$ in order that the initial boundary value problem

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0, \quad u(0) = u_0, \quad \frac{\partial u}{\partial t}(0) = u_1, \quad B_{2N+1}u = 0 \tag{3.6}$$

be well posed. These conditions are

$$0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_N < 1, \quad \beta_k > 0\ (1 \le k \le N), \quad \sum_{k=1}^{N}\frac{\beta_k}{1-\alpha_k} < \gamma. \tag{3.7}$$

Our goal in this section is to check that the same result can be obtained (and even improved in some sense) using energy techniques, as we did for the second-order and third-order boundary conditions in §§2.2 and 2.3. For technical reasons we shall divide our analysis into two parts considered separately:

(i) *Boundary conditions of even order.* These correspond to $\alpha_1 = 0 < \alpha_2$. We treat these conditions in §3.2. As we shall see, the second-order boundary condition (2.19) treated in §2.2.2 can be considered as a model for these conditions.

(ii) *Boundary conditions of odd order.* These correspond to $\alpha_1 > 0$ and are treated in §3.1. The third-order boundary condition (2.47) is the model for these conditions.

The distinction between these two cases comes from the fact that in the case $\alpha_1 = 0$, the operator B_{2N+1}, which is of order $2N+1$, is a multiple of $\frac{\partial}{\partial t}$, that is, $B_{2N+1} = \frac{\partial}{\partial t}(\widetilde{B}_{2N})$. Therefore, if we integrate (3.4) once in time (assuming that the initial data vanish on the boundary), our boundary condition can be rewritten $\widetilde{B}_{2N}u = 0$, where now $\widetilde{B}_{2N}$ is an operator of order $2N$.

For technical reasons, it seemed more natural to us to first treat the odd conditions. However, in [7], we have adopted another approach to obtain the same conclusions as here. In particular, a link between the even and the odd conditions is presented in that paper.

3.1. **Odd-order boundary conditions.** The first step of the analysis is to write a new boundary condition which is equivalent to (3.4) for any smooth solution of the wave equation. For this, it suffices to replace $\partial^2/\partial x^2$ formally by $\partial^2/\partial t^2 - \partial^2/\partial y^2$ in (3.4). In this manner, one proves that any C^∞ function u satisfying (3.4) on the boundary $\Gamma = \partial\Omega$, together with the wave equation (2.1) in the domain Ω also satisfies

$$B^*_{2N+1}u = B^*_{2N+1}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial y}\right)u = 0, \tag{3.8}$$

where the linear differential operator B^*_{2N+1} is given by
(3.9)

$$B^*_{2N+1}u = B^*_{2N+1}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial y}\right) = \frac{\partial Q}{\partial y} + \gamma\frac{\partial}{\partial t}Q - \frac{\partial}{\partial t}\sum_{k=1}^{N}\beta_k\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\right)\cdot Q_k,$$

where

$$Q = Q\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial y}\right) = \prod_{k=1}^{N} P_k, \qquad Q_k = Q_k\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial y}\right) = \prod_{j\neq k} P_j,$$

$$P_j = P_k\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial y}\right) = \alpha_j\frac{\partial^2}{\partial y^2} + (1-\alpha_j)\frac{\partial^2}{\partial t^2}.$$

In order to single out the principal part of B^*_{2N+1} with respect to time, we note that

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} = \frac{1}{1-\alpha_k}\left\{P_k - \frac{\partial^2}{\partial y^2}\right\},$$

so that we can also write

$$B^*_{2N+1} = \frac{\partial Q}{\partial y} + \left[\gamma - \sum_{k=1}^{N}\frac{\beta_k}{1-\alpha_k}\right]\frac{\partial Q}{\partial t} + \sum_{k=1}^{N}\frac{\beta_k}{1-\alpha_k}\frac{\partial^3 Q_k}{\partial y^2\,\partial t}.$$

Note that this expression is linked to the following expression for the rational function $r(s)$:

$$r(s) = \left(\gamma - \sum_{k=1}^{N}\frac{\beta_k}{1-\alpha_k}\right) - \sum_{k=1}^{N}\frac{\beta_k}{\alpha_k}\frac{1}{1-\alpha_k s^2}.$$

Therefore, the boundary condition (3.8) can be rewritten as

$$-\frac{\partial}{\partial y}(Qu) = \left[\gamma - \sum_{k=1}^{N}\frac{\beta_k}{1-\alpha_k}\right]\frac{\partial Qu}{\partial t} + \sum_{k=1}^{N}\frac{\beta_k}{1-\alpha_k}\frac{\partial^3}{\partial y^2\,\partial t}Q_k u. \tag{3.10}$$

Now, we turn to the derivation of our main energy estimate. We first introduce $v = Qu$. Applying (2.4) to v leads to

$$\frac{d}{dt}\{E(v)\} = \int_\Gamma \frac{\partial Qu}{\partial y}\frac{\partial Qu}{\partial t}\,dx. \tag{3.11}$$

Now, for $1 \le k \le N$, we introduce the functions $v_k = Q_k\partial^2 u/\partial y^2$ and $w_k = Q_k\partial^2 u/\partial y\,\partial t$ and apply (2.4) to v_k and w_k. We obtain

$$\frac{d}{dt}\{E(v_k)\} = \int_\Gamma Q_k\frac{\partial^3 u}{\partial y^3}\cdot Q_k\frac{\partial^3 u}{\partial y^2\,\partial t}\,dx, \tag{3.12}$$

$$\frac{d}{dt}\{E(w_k)\} = \int_\Gamma Q_k\frac{\partial^3 u}{\partial y\,\partial t^2}\cdot Q_k\frac{\partial^3 u}{\partial y^2\,\partial t}\,dx. \tag{3.13}$$

We multiply (3.12) by α_k and (3.13) by $(1-\alpha_k)$ and add the two resulting equalities. Using the fact that $\alpha_k\partial^2/\partial y^2 + (1-\alpha_k)\partial^2/\partial t^2 = P_k$ and that $P_k Q_k = Q$, we obtain

$$\frac{d}{dt}\{\alpha_k E(v_k) + (1-\alpha_k)E(w_k)\} = \int_\Gamma Q_k \frac{\partial^3 u}{\partial y^2 \partial t} \cdot \frac{\partial Qu}{\partial y}\, dx. \tag{3.14}$$

Now we multiply (3.14) by $\beta_k/(1-\alpha_k)$, sum over k, and add the result to (3.12) multiplied by $(\gamma - \sum_{k=1}^N \beta_k/(1-\alpha_k))$. This leads to

(3.15)

$$\frac{d}{dt}\left\{\left(\gamma - \sum_{k=1}^N \frac{\beta_k}{1-\alpha_k}\right) E(v) + \sum_{k=1}^N \frac{\beta_k \alpha_k}{1-\alpha_k} E(v_k) + \sum_{k=1}^N \beta_k E(w_k)\right\}$$
$$= \int_\Gamma \frac{\partial Qu}{\partial y}\left(\left(\gamma - \sum_{k=1}^N \frac{\beta_k}{1-\alpha_k}\right)\frac{\partial Qu}{\partial t} + \sum_{k=1}^N \frac{\beta_k}{1-\alpha_k}\frac{\partial^3}{\partial y^2 \partial t} Q_k u\right) dx.$$

Using now (3.10), we finally get

$$\frac{d}{dt}\{E_{2N+1}(u)\} = -\int_\Gamma \left|\frac{\partial}{\partial y}(Qu)\right|^2 dx < 0, \tag{3.16}$$

where $E_{2N+1}(u)$ denotes the $(2N+1)$st-order energy

$$\begin{aligned} E_{2N+1}(u) = &\left(\gamma - \sum_{k=1}^N \frac{\beta_k}{1-\alpha_k}\right) E(Qu) + \sum_{k=1}^N \frac{\beta_k \alpha_k}{1-\alpha_k} E\left(Q_k \frac{\partial^2 u}{\partial y^2}\right) \\ &+ \sum_{k=1}^N \beta_k E\left(Q_k \frac{\partial^2 u}{\partial y \partial t}\right). \end{aligned} \tag{3.17}$$

If the conditions (3.7) are satisfied, all the coefficients appearing in $E_{2N+1}(u)$ are strictly positive, and the fact that $E_{2N+1}(u)$ is a decreasing function of time leads to uniform bounds (by $C(\|u_0\|_{H^{2N+1}(\Omega)} + \|u_1\|_{H^{2N}(\Omega)})$) in time of the L^2-norms in space of the quantities

$$\begin{aligned} &\frac{\partial}{\partial t}(Qu),\ \frac{\partial}{\partial x}(Qu),\ \frac{\partial}{\partial y}(Qu), \\ &\frac{\partial}{\partial t}\left(Q_k \frac{\partial^2 u}{\partial y^2}\right),\ \frac{\partial}{\partial x}\left(Q_k \frac{\partial^2 u}{\partial y^2}\right),\ \frac{\partial}{\partial y}\left(Q_k \frac{\partial^2 u}{\partial y^2}\right) \quad (1 \le k \le N), \\ &\frac{\partial}{\partial t}\left(Q_k \frac{\partial^2 u}{\partial y \partial t}\right),\ \frac{\partial}{\partial x}\left(Q_k \frac{\partial^2 u}{\partial y \partial t}\right),\ \frac{\partial}{\partial y}\left(Q_k \frac{\partial^2 u}{\partial y \partial t}\right) \quad (1 \le k \le N). \end{aligned} \tag{3.18}$$

Now, let $\mathscr{P}$ denote the set of polynomials of two variables, $P(s, \xi)$, which are homogeneous and of degree $2N$. The set $\mathscr{P}$ is a vector space of dimension $2N+1$, and it is easy to check that it is generated by the $2N+1$ polynomials

$$Q(s, \xi),\ s^2 Q_k(s, \xi)\ (1 \le k \le N),\ s\xi Q_k(x, \xi)\ (1 \le k \le N). \tag{3.19}$$

Indeed, assume that there exists $(\lambda, \lambda_k, \mu_k)$ such that

$$\lambda Q(s, \xi) + \sum_{k=1}^N \{\lambda_k s^2 Q_k(s, \xi) + \mu_k s\xi Q_k(s, \xi)\} = 0. \tag{3.20}$$

Choosing $s = \pm(\alpha_k/(1-\alpha_k))^{1/2}\xi$ and using the fact that all the α_k are distinct, so that $Q_k(\xi, \pm(\alpha_k/(1-\alpha_k))^{1/2}\xi) \neq 0$, we obtain for each value of k the equalities

$$\mu_k \pm \left(\frac{\alpha_k}{1-\alpha_k}\right)^{1/2} \lambda_k = 0,$$

which imply that $\lambda_k = \mu_k = 0$. Returning to (3.20), we deduce that $\lambda = 0$. This proves that the $2N+1$ polynomials given by (3.19) are linearly independent and thus constitute a basis of $\mathscr{P}$. From (3.18) we deduce that the quantities

$$\frac{\partial}{\partial t}\left(P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial y}\right)u\right), \quad \frac{\partial}{\partial x}\left(P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial y}\right)u\right), \quad \frac{\partial}{\partial y}\left(P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial y}\right)u\right), \tag{3.21}$$

are bounded in $L^\infty(\mathbb{R}^+; L^2(\Omega))$ by $C(\|u_0\|_{H^{2N+1}(\Omega)} + \|u_1\|_{H^{2N}(\Omega)})$ for any $P(\cdot, \cdot)$ in $\mathscr{P}$. This implies that

$$\begin{aligned} &\forall \alpha \in \mathbb{N}^3/|\alpha| = 2N+1, \\ &\|D^\alpha u\|_{L^\infty(\mathbb{R}^+; L^2(\Omega))} \le C(\|u_0\|_{H^{2N+1}(\Omega)} + \|u_1\|_{H^{2N}(\Omega)}). \end{aligned} \tag{3.22}$$

Indeed, any $D^\alpha u$ containing at most a first-order derivative with respect to x can be written in the form (3.21). If derivatives of higher order appear, one can reduce to the preceding case by (repeated) use of the wave equation.

Estimates (3.22) are nothing but a strong stability result in the sense of Kreiss. Note that we can also derive boundary estimates from (3.17). For instance, (3.17) directly implies that $\frac{\partial}{\partial y}(Qu)$ can be estimated in $L^2(\mathbb{R}^+; L^2(\Gamma))$.

Once again, we see that for the $(2N+1)$st-order boundary condition (3.8), the energy one bounds is a $(2N+1)$st-order energy, in the sense defined in §1, which means that it involves $(2N+1)$st-order derivatives of the solution. This is consistent with the estimates one would obtain directly by an application of the Kreiss theory. Indeed, if one wants to put the initial boundary problem for the wave equation, coupled with condition (3.8), in the form of the first-order systems studied by Kreiss, one has to introduce an unknown vector function U of dimension $2N+1$ whose coordinates are $(2N+1)$st-order derivatives of the function u. Therefore, the L^2-norm in the space of U corresponds to some energy of order $2N+1$. Let us summarize the results of this section in the following theorem:

Theorem 3.1. *Under the condition* (3.7), *the initial boundary value problem* (2.1), (3.4) *is strongly well posed in the sense of Kreiss, and the unique solution satisfies the identity* (3.16), *which yields the decay in time of the* $(2N+1)$*st-order energy* E_{2N+1} *defined by* (3.17). *Moreover, this solution satisfies the a priori estimate*

$$\sup_{|\alpha|=2N+1} \|D^\alpha u\|_{L^\infty(\mathbb{R}^+; L^2(\Omega))} \le C(\|u_0\|_{H^{2N+1}(\Omega)} + \|u_1\|_{H^{2N}(\Omega)}),$$

where the positive constant C *depends only on* γ, α_k, *and* β_k, $1 \le k \le N$.

3.2. **Even-order boundary conditions.** As we said before, the even-order boundary conditions correspond to the particular case $\alpha_1 = 0$, so that the a priori estimates we obtained in §3.1, in particular the decay in time of the $(2N+1)$st-order $E_{2N+1}(u)$ given by (3.17) is still valid when we take $\alpha_1 = 0$. Nevertheless, we prefer to treat the even-order conditions separately because, since condition (3.4) for $\alpha_1 = 0$ corresponds to a $(2N)$th-order condition, one can expect the decay in time of a $(2N)$th-order energy instead of a $(2N+1)$st-order one as is $E_{2N+1}(u)$. Moreover, this is more consistent with the definition of strong well-posedness by Kreiss.

Therefore, the treatment of the conditions of even order will be slightly different from the one of the conditions of odd order. First we note that, by similar arguments as in §3.1, condition (3.4) is equivalent to

$$B_{2N}^* u = B_{2N}^* \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial y}\right) u = 0 \tag{3.23}$$

(we have integrated (3.4) once in time, which is possible since $\alpha_1 = 0$, and replaced $\partial^2/\partial x^2$ by $\partial^2/\partial t^2 - \partial^2/\partial y^2$), where the differential operator B_{2N}^* is given by

$$B_{2N}^* = \left[\frac{\partial^2}{\partial y\,\partial t} + \gamma\frac{\partial^2}{\partial t^2}\right] Q_1 - \sum_{k=1}^{N} \beta_k \left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}\right] Q_k, \tag{3.24}$$

with the differential operators Q_k defined in the previous section (simply note that in our case, $P_1 = \partial^2/\partial t^2$). Using the identities

$$P_k Q_k = P_1 Q_1 = \frac{\partial^2 Q_1}{\partial t^2}, \qquad \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} = \frac{1}{1-\alpha_k}\left[P_k - \frac{\partial^2}{\partial y^2}\right],$$

we deduce that

$$B_{2N}^* = \left(\gamma - \sum_{k=1}^{N} \frac{\beta_k}{1-\alpha_k}\right) \frac{\partial^2 Q_1}{\partial t^2} + \frac{\partial^2 Q_1}{\partial y\,\partial t} + \sum_{k=1}^{N} \frac{\beta_k}{1-\alpha_k}\frac{\partial^2 Q_k}{\partial y^2}. \tag{3.25}$$

Therefore, condition (3.23) can be rewritten as

$$\left(\gamma - \sum_{k=1}^{N} \frac{\beta_k}{1-\alpha_k}\right) \frac{\partial^2}{\partial t^2} Q_1 u + \beta_1 \frac{\partial^2}{\partial y^2} Q_1 u + \sum_{k=2}^{N} \frac{\beta_k}{1-\alpha_k}\frac{\partial^2 Q_k}{\partial y^2} = -\frac{\partial^2 Q_1}{\partial y\,\partial t}. \tag{3.26}$$

Now consider first the functions $v_1 = \partial(Q_1 u)/\partial t$ and $w_1 = \partial(Q_1 u)/\partial y$. They satisfy

$$\frac{d}{dt}\{E(v_1)\} = \int_\Gamma \frac{\partial^2 Q_1 u}{\partial y\,\partial t}\frac{\partial^2 Q_1 u}{\partial t^2}\,dx, \tag{3.27}$$

$$\frac{d}{dt}\{E(w_1)\} = \int_\Gamma \frac{\partial^2 Q_1 u}{\partial y\,\partial t}\frac{\partial^2 Q_1 u}{\partial y^2}\,dx. \tag{3.28}$$

Introducing the operators $q_k = \prod_{j\neq 1,k} P_k$ for $2 \le k \le N$, we define the functions $v_k = \partial^3(q_k u)/\partial y^2 \partial t$ and $w_k = \partial^3(q_k u)/\partial t^2\,\partial y$ and note that

$$\frac{\partial v_k}{\partial t} = \frac{\partial w_k}{\partial y} = \frac{\partial^4}{\partial y^2 \partial t^2} q_k u = \frac{\partial^2}{\partial y^2} Q_k u, \tag{3.29}$$

and that

$$
\begin{aligned}
\frac{\partial v_k}{\partial y} &= \frac{\partial^4}{\partial y^3\,\partial t} q_k u = \frac{\partial^2}{\partial y\,\partial t}\left[\frac{\partial^2}{\partial y^2} q_k u\right], \\
\frac{\partial w_k}{\partial t} &= \frac{\partial^4}{\partial y\,\partial t^3} q_k u = \frac{\partial^2}{\partial y\,\partial t}\left[\frac{\partial^2}{\partial t^2} q_k u\right].
\end{aligned} \tag{3.30}
$$

Applying identity (2.4) to v_k and w_k, using (3.29) and (3.30), we get

$$
\frac{d}{dt}E(v_k) = \int_\Gamma \frac{\partial^2}{\partial y^2} Q_k u \cdot \frac{\partial^2}{\partial y\,\partial t}\left[\frac{\partial^2}{\partial y^2} q_k u\right] dx, \tag{3.31}
$$

$$
\frac{d}{dt}E(w_k) = \int_\Gamma \frac{\partial^2}{\partial y^2} Q_k u \cdot \frac{\partial^2}{\partial y\,\partial t}\left[\frac{\partial^2}{\partial t^2} q_k u\right] dx. \tag{3.32}
$$

Now we note that by construction, $P_k q_k = Q_1$, so that, if we multiply (3.31) by α_k and (3.32) by $1-\alpha_k$, we obtain

$$
\frac{d}{dt}\{\alpha_k E(v_k) + (1-\alpha_k)E(w_k)\} = \int_\Gamma \frac{\partial^2}{\partial y^2} Q_k u \cdot \frac{\partial^2}{\partial y\,\partial t} Q_1 u\,dx. \tag{3.33}
$$

Multiply (3.27) by $\gamma - \sum_{k=1}^N \beta_k/(1-\alpha_k)$, (3.28) by β_1, and (3.33) by $\beta_k/(1-\alpha_k)$ and sum all these equalities to get

$$
\begin{aligned}
&\frac{d}{dt}\left\{\left(\gamma - \sum_{k=1}^N \frac{\beta_k}{1-\alpha_k}\right) E(v_1) + \beta_1 E(w_1) + \sum_{k=2}^N \left[\frac{\beta_k\alpha_k}{1-\alpha_k} E(v_k) + \beta_k E(w_k)\right]\right\} \\
&\quad = \int_\Gamma \frac{\partial^2}{\partial y\,\partial t}(Q_1 u) \\
&\qquad \times \left[\left(\gamma - \sum_{k=1}^N \frac{\beta_k}{1-\alpha_k}\right)\frac{\partial^2}{\partial t^2} Q_1 u + \beta_1 \frac{\partial^2}{\partial y^2} Q_1 u + \sum_{k=2}^N \frac{\beta_k}{1-\alpha_k}\frac{\partial^2}{\partial y^2} Q_k u\right] dx.
\end{aligned}
$$

By (3.26), we finally obtain

$$
\frac{d}{dt}\{E_{2N}(u)\} = -\int_\Gamma \left|\frac{\partial^2}{\partial y\,\partial t}(Q_1 u)\right|^2 dx, \tag{3.34}
$$

where $E_{2N}(u)$ denotes the $(2N)$th-order energy

$$
\begin{aligned}
E_{2N}(u) = {} & \left(\gamma - \sum_{k=1}^N \frac{\beta_k}{1-\alpha_k}\right) E\left(\frac{\partial}{\partial t} Q_1 u\right) + \beta_1 E\left(\frac{\partial}{\partial y} Q_1 u\right) \\
& + \sum_{k=2}^N \left\{\frac{\beta_k\alpha_k}{1-\alpha_k} E\left(\frac{\partial^3}{\partial y^2\,\partial t} q_k u\right) + \beta_k E\left(\frac{\partial^3}{\partial y\,\partial t^2} q_k u\right)\right\}.
\end{aligned} \tag{3.35}
$$

It is then easy to check that, as in §3.1, (3.34) is equivalent to a strong stability result if the stability conditions (3.7) are satisfied. In fact, it suffices to verify that the $2N$ operators $\frac{\partial}{\partial t} Q_1$, $\frac{\partial}{\partial y} Q_1$ and $\{(\partial^3/\partial y^2\,\partial t)q_k, (\partial^3/\partial y\,\partial t^2)q_k, 2 \le k \le N\}$ generate all the homogeneous differential operators of order $2N-1$ with respect to y and t. The details are left to the reader. One finally obtains

Theorem 3.2. *Under the condition* (3.7), *the initial boundary value problem* (2.1), (3.23) *is strongly well posed in the sense of Kreiss, and the unique solution satisfies the identity* (3.34), *which yields the decay in time of the* $(2N)$*th-order energy* E_{2N} *defined by* (3.35). *Moreover, this solution satisfies the a priori estimate*

$$\sup_{|\alpha|=2N} \|D^\alpha u\|_{L^\infty(\mathbb{R}^+;\, L^2(\Omega))} \leq C(\|u_0\|_{H^{2N}(\Omega)} + \|u_1\|_{H^{2N-1}(\Omega)}),$$

where the positive constant C *depends only on* γ, α_k, *and* β_k, $1 \leq k \leq N$.

4. Conclusion

In this paper, we have revisited the theory of strong well-posedness of initial boundary value problems for the wave equation via the approach of energy estimates, which is, to our knowledge, new. This method allowed us to find again the conditions obtained by Trefethen and Halpern in the framework of the modal analysis and to improve the a priori estimates directly deduced from the theory of Kreiss.

Moreover, a second interest of our approach lies in the fact that we are able to extend some strong stability results to the case of variable coefficients, even if these coefficients are not smooth. This will be the topic of a forthcoming paper.

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